Prof. Rüdiger Frey
Institute for Statistics and Mathmatics
WU Vienna
ruediger.frey@wu.ac.at

**Master QFin, CTFI**
**Final Exam, Wed 22.6. 12.00-13.30**

**Hints**

- This is a closed-book exam.
- Good luck !

**1. Ornstein Uhlenbeck process and Feynman Kac** Consider the solution $X$ of the SDE $dX_t = -\kappa X_t dt + dW_t$, $X_0 = x \in \mathbb{R}$ for a standard Brownian motion $W$ and some $\kappa > 0$.

a) (4 points)Show that the process $Y$ with $Y_t = e^{\kappa t} X_t$ has dynamics $dY_t = e^{\kappa t} dW_t$. Conclude that $Y_t = x + \int_0^t e^{\kappa s} dW_s$ and hence

$$X_t = e^{-\kappa t} x + e^{-\kappa t} \int_0^t e^{\kappa s} dW_s.$$

Use this to compute $E(X_t)$.

b) (3 points) Use the Feynman Kac formula to solve the terminal value problems.

$$f_t(t,x) - x f_x(t,x) + \frac{1}{2} f_{xx}(t,x) = 0, \quad (t,x) \in [0,T) \times \mathbb{R},$$

with terminal condition $f(T,x) = x$. Hint: use a) for the computation.

**2. Black Scholes model.** Consider the Black Scholes model with stock price dynamics $dS_t = \mu S_t dt + \sigma S_t dW_t$, initial stock price $S_0 > 0$ and with money market account $B_t = \exp(rt)$ for $r > 0$.

a) (4 points) Consider the fair price $V_t = V(t, S_t)$ of a terminal value claim with payoff $h(S_T)$. Show that the function $V(t,S)$ solves the terminal value problem

$$V_t(t,S) + rSV_S(t,S) + \frac{1}{2}\sigma^2 S^2 V_{SS}(t,S) = rV(t,S), \quad (t,S) \in [0,T) \times \mathbb{R}^+, \qquad (1)$$

with terminal condition $V(T,S) = h(S)$. Use Feynman Kac to derive a risk-neutral pricing formula for $V(t,S)$.

b) (3 points) Consider a so-called power option with $h(S) = S^2$. Use the risk neutral pricing formula to show that $V(0,S) = S^2 \exp\left((r+\sigma^2)T\right)$ and compute the Delta of the option. Hint: For a rv $X = \exp(Z)$ with $Z \sim N(\mu, \beta^2)$ it holds that $E(X) = \exp(\mu + \frac{1}{2}\beta^2)$.

c) (2 points) Suppose that the true volatility is equal to $\tilde{\sigma} > \sigma$. Consider a trader who has sold one power option for the Black Scholes price corresponding to the volatility $\sigma$ and who replicates the option using a selffinancing strategy with stock position equal to the Delta of the option (again for volatility $\sigma$. Will the trader make a profit or a loss? Justify your answer using the result on the tracking error in the Black Scholes model.